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An obstacle problem for nonlinear hemivariational inequalities at resonance

Sophia Th. Kyritsi and Nikolaos S. Papageorgiou *

National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece

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Abstract

In this paper we examine an obstacle problem for a nonlinear hemivariational inequality at resonance driven by the p-Laplacian. Using a variational approach based on the nonsmooth critical point theory for locally Lipschitz functionals defined on a closed, convex set, we prove two existence theorems. In the second theorem we have a pointwise interpretation of the obstacle problem, assuming in addition that the obstacle is also a kind of lower solution for the nonlinear elliptic differential inclusion.

Keywords: Locally Lipschitz function; Generalized subdifferential; Nonsmooth C-condition; Critical point; Obstacle; Sobolev space; p-Laplacian; Principal eigenvalue; Rayleigh quotient; Hemivariational inequality at resonance; Lower solution

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary Γ, $\psi \in W^{1,p}(Z) \cap L^\infty(Z)$, $2 \leqslant p < \infty$. In this paper we study the following obstacle problem:

$$\left\{\begin{array}{l} \int_Z \|Dx\|^{p-2}(Dx, Dy - Dx)_{\mathbb{R}^N}\,dz - \lambda_1 \int_Z |x|^{p-2}x(y-x)\,dz \\ \quad \geqslant \int_Z u^*(y-x)\,dz \\ \quad \text{for all} \quad y \in C = \{y \in W_0^{1,p}(Z)\colon\ y(z) \geqslant \psi(z) \text{ a.e. on } Z\} \\ \quad \text{and} \quad u^* \in L^q(Z), \quad u^*(z) \in \partial j\big(z, x(z)\big) \quad \text{a.e. on } Z \end{array}\right\}. \tag{1}$$

* Corresponding author.
E-mail address: npapg@math.ntua.gr (N.S. Papageorgiou).

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Here λ_1 is a principal eigenvalue of the negative p-Laplacian with Dirichlet boundary condition (i.e., of $(-\Delta_p, W_0^{1,p}(Z)))$ and $\partial j(z,x)$ is the subdifferential in the sense of Clarke [4] of the locally Lipschitz function $x \to j(z,x)$. So problem (1) is an obstacle problem for a so-called "hemivariational inequality at resonance." Hemivariational inequalities originated in the context of continuum mechanics, when we deal with problems which have nonsmooth and nonconvex potential. Roughly speaking, mechanical problems involving nonmonotone stress-strain laws or boundary conditions derived by nonconvex superpotentials, lead to hemivariational inequalities. For concrete applications in mechanics and engineering, we refer to Naniewicz and Panagiotopoulos [17] and Panagiotopoulos [18]. We should also mention that hemivariational inequalities incorporate as a special case problems with discontinuities (see Chang [3] and Kourogenis and Papageorgiou [9]).

Various forms of obstacle problems and of general variational inequalities with smooth potential were studied in the last decade using a variety of methods. So we mention the works of Quittner [19], Le [11] where bifurcation methods are used, the works of Szulkin [23], Le and Schmitt [14] who follow a variational approach, the work of Szulkin [22] based on degree theory, the work of Ang et al. [2] based on recession arguments, the book of Showalter [20] where operators of monotone type are used and the references therein. We also mention the recent interesting works of Le [12,13], who developed the subsolution–supersolution method for a broad class of noncoercive "smooth" variational inequalities.

Our interest for problem (1) has been primarily from a mathematical viewpoint. Problem (1) is new variational expression with a nonsmooth, nonconvex potential function. In fact problem (1) can be characterized as a "variational-hemivariational inequality," since it incorporates the mathematical features of both variational and hemivariational inequalities. Moreover, our driving partial differential operator is nonlinear (the p-Laplacian), similar to the recent important works of Le [12,13]. In the past most works on variational inequalities had a linear differential operator (usually the Laplacian). Although our primary motivation was mathematical, the works of Naniewicz and Panagiotopoulos [17] and Panagiotopoulos [18] have illustrated that variational expressions similar to (1) arise in mechanics in order to deal with problems whose variational forms are such inequalities which express the principle of virtual work or power. In particular the resonant hemivariational inequalities, such as problem (1), are closely connected with the stability analysis of the corresponding mechanical system (such as the maximum loading sustained by the structure before instability occurs, for details see Panagiotopoulos [18]).

Our approach is variational and it is based on the nonsmooth critical point theory for locally Lipschitz functionals defined on a closed, convex set, which was developed recently by Kyritsi and Papageorgiou [10]. For the convenience of the reader in the next section we recall the basic aspects of this theory.

2. Mathematical preliminaries

The constrained nonsmooth critical point theory which is the abstract framework of our approach, is based on the subdifferential theory of Clarke [4] and extends the theories of Chang [3] and Kourogenis and Papageorgiou [9].

Let X be a Banach space and X^* its topological dual. A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $x \in X$ there exists a neighborhood U of x and a constant $k > 0$ depending on U such that $|\varphi(y) - \varphi(z)| \leqslant k\|y - z\|$ for all $y, z \in U$. For each $h \in X$, we define the generalized directional derivative $\varphi^0(x; h)$ by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that the function $h \to \varphi^0(x; h)$ from X into $\mathbb{R}$ is sublinear and continuous. So by the Hahn–Banach theorem $\varphi^0(x; \cdot)$ is the support function of a nonempty, convex and w^*-compact set given by

$$\partial\varphi(x) = \left\{x^* \in X^*:\ (x^*, h) \leqslant \varphi^0(x; h) \text{ for all } h \in X\right\}.$$

The set $\partial\varphi(x)$ is known as the (generalized or Clarke) subdifferential of φ at x. If $\varphi, g : X \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(\varphi + g)(x) \subseteq \partial\varphi(x) + \partial g(x)$ and $\partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$ for all $x \in X$ and all $\lambda \in \mathbb{R}$. If $\varphi : X \to \mathbb{R}$ is convex, then it is well-known from convex analysis that φ is locally Lipschitz. Then the generalized subdifferential and the subdifferential in the sense of convex analysis (see, for example, Hu and Papageorgiou [6, p. 267]) coincide, while the generalized directional derivative $\varphi^0(x; h)$ coincides with the usual directional derivative of convex analysis

$$\varphi'(x; h) = \lim_{\lambda \downarrow 0} \frac{\varphi(x + \lambda h) - \varphi(x)}{\lambda}.$$

Moreover, if φ is strictly differentiable at x (in particular if φ is continuously Gateaux differentiable at x), then $\partial\varphi(x) = \{\varphi'(x)\}$. The multifunction $\partial\varphi : X \to 2^{X^*} \setminus \{\emptyset\}$ is upper semicontinuous from X furnished with the norm topology into X^* furnished with the weak* topology (denoted by $X^*_{w^*}$). This means that for every $V \subseteq X^*$ w^*-open set, the set $\partial\phi^+(V) = \{x \in X:\ \partial\varphi(x) \subseteq V\}$ is norm open in X (see Hu and Papageorgiou [6, p. 36]). In particular this implies that $Gr\partial\varphi = \{(x, x^*) \in X \times X^*:\ x^* \in \partial\phi(x)\}$ is closed in $X \times X^*_{w^*}$ (see Hu and Papageorgiou [6, p. 41]).

Now suppose that X is a reflexive Banach space and $C \subseteq X$ a nonempty, closed and convex set. Given a locally Lipschitz function $\varphi : C \to \mathbb{R}$, for $x \in C$, we define

$$m_C(x) = \inf_{x^*} \sup_{y} \left[(x^*, x - y):\ y \in C,\ \|x - y\| < 1,\ x^* \in \partial\varphi(x)\right].$$

Here by $(\cdot\,,\cdot)$ we denote the duality brackets for the pair (X, X^*). Evidently $m_C(x) \geqslant 0$ and this quantity can be viewed as a measure of the generalized slope of φ at $x \in C$. If φ admits an extension $\hat{\varphi} \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$ and so we have

$$m_C(x) = \sup\bigl[\bigl(\varphi'(x), x-y\bigr)\colon\ y \in C,\ \|x-y\| < 1\bigr],$$

which is the quantity used by Struwe [21, p. 147] in his constrained smooth critical point theory. Moreover, if $C = X$, then we have

$$m_C(x) = m(x) = \inf\bigl[\|x^*\|\colon\ x^* \in \partial\varphi(x)\bigr],$$

which is the quantity used by Chang [3] and Kourogenis and Papageorgiou [9] in their nonsmooth versions of the critical point theory. The function m_C is lower semicontinuous and a point $x \in C$ is a "critical point" of φ on C, if $m_C(x) = 0$. The value $\beta = \varphi(x)$ is called "critical value" of φ. Note that if $C = X$, then as we already pointed out $m_C(x) = m(x) = \inf[\|x^*\|\colon\ x^* \in \partial\varphi(x)]$ and observe that this infimum is attained because the set $\partial\varphi(x)$ is w-compact and the norm functional on X^* is weakly lower semicontinuous. So $x \in X$ is a critical point of φ if and only if $0 \in \partial\varphi(x)$. This is the notion used by Chang [3] and Kourogenis and Papageorgiou [9].

As it is well-known from the smooth theory, the variational approach uses a compactness-type condition on φ. In the present nonsmooth and constrained setting, this condition takes the following form:

> The function $\varphi\colon C \to X$ satisfies the "nonsmooth C-condition on C" if any sequence $\{x_n\}_{n\geqslant 1} \subseteq C$ such that $\{\varphi(x_n)\}_{n\geqslant 1}$ is bounded and $(1 + \|x_n\|) \times m_C(x_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

Problem (1) is at resonance and so we need to say a few things about the spectrum of $(-\Delta_p, W_0^{1,p}(Z))$. So consider the following nonlinear eigenvalue problem:

$$\left\{\begin{array}{l} -\operatorname{div}\bigl(\|Dx(z)\|^{p-2}Dx(z)\bigr) = \lambda\bigl|x(z)\bigr|^{p-2}x(z) \text{ a.e. on } Z, \\ x|_\Gamma = 0 \end{array}\right\}. \tag{2}$$

The least number $\lambda \in \mathbb{R}$ for which problem (2) has a nontrivial solution is the first eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$ and it is denoted by λ_1. We know that $\lambda_1 > 0$, it is isolated and simple (i.e., the associated eigenfunctions are constant multiples of each other). Moreover, $\lambda_1 > 0$ has a variational characterization (Rayleigh quotient)

$$\lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p}\colon\ x \in W_0^{1,p}(Z),\ x \neq 0\right]. \tag{3}$$

This minimum is realized at the normalized eigenfunction u_1. Note that if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$ and so we infer that the first eigenfunction u_1 does not change sign on Z. In fact using the regularity result of Lieberman [15] we can show that $u_1 \in C^1(\overline{Z})$ and then the strong maximum principle of Vazquez [24] implies that $u_1(z) > 0$ for all $z \in Z$. For details we refer to Anane [1] and Lindqvist [16]. The Liusternik–Schnirelmann theory gives, in addition to $\lambda_1 > 0$, a whole strictly increasing sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, $\lambda_n \to +\infty$, known as the "Liusternik–Schnirelmann or variational eigenvalues" of $(-\Delta_p, W_0^{1,p}(Z))$. If $p = 2$, these are all the eigenvalues, but for $p \neq 2$ we do not know if this the case.

Finally if Y is a Banach space and $E \subseteq Y$, then the "indicator function" of E, δ_E, is defined by

$$\delta_E(x) = \begin{cases} 0 & \text{if } x \in E, \\ +\infty & \text{otherwise.} \end{cases}$$

If E is nonempty, closed, convex, then δ_E is proper (i.e., it is not identically $+\infty$), lower semicontinuous and convex. Also the "support function" $\sigma(\cdot\,; E) : Y^* \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is defined by $\sigma(y^*; E) = \sup[(y^*, y)\colon\ y \in E]$. The function $y^* \to \sigma(y^*; E)$ is w^*-lower semicontinuous and sublinear and is the convex conjugate of δ_E. The convex subdifferential of δ_E at $y \in E$ is given by

$$\partial\delta_E(y) = \left\{y^* \in Y^*\colon\ (y^*, z - y) \leqslant \delta_E(z) \text{ for all } z \in Y\right\}.$$

Evidently

$$\begin{aligned} \partial\delta_E(y) = N_E(y) &= \left\{y^* \in Y^*\colon\ (y^*, y - z) \geqslant 0 \text{ for all } z \in E\right\} \\ &= \text{the normal cone to } E \text{ at } y. \end{aligned}$$

For details see Hu and Papageorgiou [6].

3. Auxiliary results

In this section we prove some auxiliary results, which will be used in Section 4 to prove two existence theorems. We impose the following conditions on the nonsmooth potential function $j(z, x)$:

$H(j)$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot\,, 0) \in L^\infty(Z)$ and

(i) for all $x \in \mathbb{R}$, $z \to j(z, x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z, x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u^* \in \partial j(z, x)$, we have

$$|u^*| \leqslant \alpha(z) + c|x|^{p-1} \quad \text{with } \alpha \in L^\infty(Z),\ c > 0;$$

(iv) there exists $\beta \in L^\infty(Z)$, $\beta(z) \leqslant 0$ a.e. on Z with strict inequality on a set of positive measure that $\limsup_{x \to +\infty} u^*/x^{p-1} \leqslant \beta(z)$ uniformly for almost all $z \in Z$ and all $u^* \in \partial j(z, x)$;

(v) there exists $g \in L^1(Z)$ such that for almost all $z \in Z$ and all $x \geqslant \psi(z)$, $j(z,x) \leqslant g(z)$.

An alternative set of hypotheses for the nonsmooth potential $j(z,x)$ is the following:

$H(j)'$: $j: Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot,0) \in L^1(Z)$ and

(i) for every $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u^* \in \partial j(z,x)$, we have

$$|u^*| \leqslant \alpha(z) + c|x|^{p-1} \quad \text{with } \alpha \in L^q(Z), \ \left(\frac{1}{p} + \frac{1}{q} = 1\right),$$

$c > 0$;

(iv) $\limsup_{x \to +\infty} u^*/x^{p-1} = 0$ uniformly for almost all $z \in Z$ and all $u^* \in \partial j(z,x)$ and if $\{x_n\}_{n \geqslant 1} \subseteq W_0^{1,p}(Z)$ is a sequence such that $x_n(z) \to +\infty$ a.e. on Z, then we have

$$\int_Z j\big(z, x_n(z)\big)\, dz \to -\infty \quad \text{as } n \to +\infty;$$

(v) there exists $g \in L^1(Z)$ such that for almost all $z \in Z$ and all $x \geqslant \psi(z)$, $j(z,x) \leqslant g(z)$.

We start with a simple observation about Sobolev functions. Recall that in general $W_0^{1,p}(Z)$ is not a Banach algebra.

Lemma 1. *If $x, y \in W^{1,p}(Z) \cap L^\infty(Z)$, then $xy \in W^{1,p}(Z)$ and $D(xy) = xDy + yDx$ (product rule).*

Proof. Let $M = \max\{\|x\|_\infty, \|y\|_\infty\}$. We know that there exist two sequences $\{\bar\theta_n\}_{n \geqslant 1}, \{\bar\xi_n\}_{n \geqslant 1} \subseteq C^\infty(\overline{Z})$ such that $\bar\theta_n \to x$, $\bar\xi_n \to y$ in $W^{1,p}$ and $\bar\theta_n(z) \to x(z)$, $\bar\xi_n(z) \to y(z)$ a.e. on Z as $n \to \infty$. We define the following truncations of $\bar\theta_n, \bar\xi_n$:

$$\theta_n(z) = \begin{cases} M & \text{if } \bar\theta_n(z) > M, \\ \bar\theta_n(z) & \text{if } -M \leqslant \theta_n(z) \leqslant M, \\ -M & \text{if } \bar\theta_n(z) < -M, \end{cases} \quad \text{and}$$

$$\xi_n(z) = \begin{cases} M & \text{if } \bar\xi_n(z) > M, \\ \bar\xi_n(z) & \text{if } -M \leqslant \xi_n(z) \leqslant M, \\ -M & \text{if } \bar\xi_n(z) < -M. \end{cases}$$

Evidently θ_n and ξ_n are locally Lipschitz functions and then so is $\theta_n\xi_n$, $n \geqslant 1$ (see Clarke [4, p. 48]). By Rademacher's theorem θ_n, ξ_n and $\theta_n\xi_n$ are differentiable almost everywhere and by the product rule we have $D(\theta_n\xi_n) = \theta_n D\xi_n + \xi_n D\theta_n \in L^p(Z, \mathbb{R}^N)$. Therefore it follows that $\theta_n\xi_n \in W^{1,p}(Z)$ for all $n \geqslant 1$. Also we have that

$$\theta_n \to x,\ \xi_n \to y \quad \text{in } W^{1,p}(Z) \quad \text{and}$$
$$\theta_n(z) \to x(z),\ \xi_n(z) \to y(z) \quad \text{a.e. on } Z \text{ as } n \to \infty.$$

Note that $\|\theta_n\|_\infty, \|\xi_n\|_\infty \leqslant M$ for all $n \geqslant 1$. Then

$$\int_Z |\theta_n\xi_n - xy|^p\,dz \leqslant 2^{p-1}\left(\int_Z |\theta_n|^p|\xi_n - y|^p\,dz + \int_Z |y|^p|\theta_n - x|^p\,dz\right) \to 0 \quad \text{as } n \to \infty.$$

Also we have

$$\begin{aligned}
&\int_Z \big\|(\theta_n D\xi_n + \xi_n D\theta_n) - (xDy + yDx)\big\|^p\,dz \\
&\quad\leqslant 2^{p-1}\left(\int_Z \|\theta_n D\xi_n - xDy\|^p\,dz + \int_Z \|\xi_n D\theta_n - yDx\|^p\,dz\right) \\
&\quad\leqslant \big(2^{p-1}\big)^2\left(\int_Z |\theta_n|^p\|D\xi_n - Dy\|^p\,dz + \int_Z \|Dy\|^p|\theta_n - x|^p\,dz\right. \\
&\qquad\left. + \int_Z |\xi_n|^p\|D\theta_n - Dx\|^p\,dz + \int_Z \|Dx\|^p|\xi_n - y|^p\,dz\right) \to 0.
\end{aligned}$$

The convergence follows from the dominated convergence theorem. So $D(\theta_n\xi_n) \to xDy + yDx$ in $L^p(Z, \mathbb{R}^N)$. But from the definition of the distributional derivative, for every $\eta \in C_0^\infty(Z)$ we have

$$\int_Z D(\theta_n\xi_n)\eta\,dz = -\int_Z (\theta_n\xi_n)D\eta\,dz.$$

Passing to the limit as $n \to \infty$, we obtain

$$\int_Z (xDy + yDx)\eta\,dz = -\int_Z (xy)D\eta\,dz.$$

Since $\eta \in C_0^\infty(Z)$ is arbitrary it follows that $xDy + yDx = D(xy) \in L^p(Z, \mathbb{R}^N)$ and so $xy \in W^{1,p}(Z)$ and the product rule holds. $\square$

Remark. If $x, y \in W_0^{1,p}(Z) \cap L^\infty(Z)$, then $xy \in W_0^{1,p}(Z)$ and the product rule holds. The proof is the same, only now the approximating sequences $\{\bar{\theta}_n\}_{n\geqslant 1}, \{\bar{\xi}_n\}_{n\geqslant 1}$ belong in $C_0^\infty(Z)$ and also θ_n and ξ_n are locally Lipschitz and of compact support. Then $\theta_n\xi_n$ is locally Lipschitz of compact support, $\theta_n\xi_n \in L^p(Z)$ and $D(\theta_n\xi_n) \in L^p(Z, \mathbb{R}^N)$. So from Theorem 2.2.6, of Kesavan [8, p. 61] we have that $\theta_n\xi_n \in W_0^{1,p}(Z)$.

Now consider the functional $\varphi : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|^p - \int_Z j\big(z, x(z)\big)\,dz.$$

We know that φ is locally Lipschitz (see Chang [3] and Hu and Papageorgiou [7, p. 313]). Also let $C \subseteq W_0^{1,p}(Z)$ be the nonempty, closed and convex set defined by

$$C = \big\{x \in W_0^{1,p}(Z):\ x(z) \geqslant \psi(z) \text{ a.e. on } Z\big\}.$$

Proposition 2. *If hypotheses $H(j)$(i) $\to$ (iv) or $H(j)'$(i) $\to$ (iv) hold and $\psi \in W^{1,p}(Z) \cap L^\infty(Z)$ with $\psi|_\Gamma \leqslant 0$, then φ satisfies the nonsmooth C-condition on C.*

Proof. Suppose that $\{x_n\}_{n\geqslant 1} \subseteq C$ is a sequence such that

$$\begin{aligned} &\big|\varphi(x_n)\big| \leqslant M_1 \quad \text{for all } n \geqslant 1 \quad \text{and} \\ &\big(1 + \|x_n\|\big)m_C(x_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4}$$

Let $C_1(x_n) = (x_n - C) \cap B_1$ with $B_1 = \{x \in W_0^{1,p}(Z):\ \|x\| < 1\}$. Since the support function $\sigma(\cdot\,; C_1(x_n))$ is weakly lower semicontinuous, $\partial\varphi(x_n)$ is weakly compact and

$$m_C(x_n) = \inf\big[\sigma\big(x_n^*; C_1(x_n)\big), x_n^* \in \partial\varphi(x_n)\big]$$

from the Weierstrass theorem it follows that there exists $x_n^* \in \partial\varphi(x_n)$ such that $m_C(x_n) = \sigma(x_n^*; C_1(x_n)), n \geqslant 1$. If by $\langle\cdot\,,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))(1/p + 1/q = 1)$, for every $u \in C$ with $\|x_n - u\| < 1$ we have

$$\begin{aligned} \big(1 + \|x_n\|\big)\big\langle x_n^*, x_n - u\big\rangle &\leqslant \big(1 + \|x_n\|\big)\sigma\big(x_n^*, C_1(x_n)\big) \\ &\leqslant \varepsilon_n \quad \text{with } \varepsilon_n \downarrow 0 \quad \big(\text{see (4)}\big). \end{aligned} \tag{5}$$

We claim that $\{x_n\}_{n\geqslant 1} \subseteq C$ is bounded in $W_0^{1,p}(Z)$. Suppose that this is not true. Then by passing to a subsequence if necessary, we may assume that $\|x_n\| \to +\infty$ as $n \to \infty$. Set $y_n = x_n/\|x_n\|$, $n \geqslant 1$. Evidently we may assume that $y_n \xrightarrow{w} y$ in $W_0^{1,p}(Z)$ and $y_n \to y$ in $L^p(Z)$ as $n \to \infty$ (recall that $W_0^{1,p}(Z)$

is embedded compactly in $L^p(Z)$). For every $n \geqslant 1$ we have $\psi \leqslant x_n$ and so $\psi/\|x_n\| \leqslant y_n, n \geqslant 1$. Passing to the limit we see that $y \geqslant 0$.

Next consider the following nondecreasing Lipschitz continuous function defined by

$$\gamma_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leqslant \delta(\varepsilon), \\ \int_{\delta(\varepsilon)}^t & \text{if } \delta(\varepsilon) \leqslant t \leqslant \varepsilon, \varepsilon > 0, \\ 1 & \text{if } \varepsilon \leqslant t. \end{cases}$$

In this definition given $\varepsilon > 0$, we choose $\delta(\varepsilon) \in (0, \varepsilon)$ such that $\int_{\delta(\varepsilon)}^{\varepsilon} \frac{ds}{s} = 1$. Note that for all $t \in \mathbb{R}$ $\gamma_\varepsilon(t) \to \chi_{\dot{\mathbb{R}}_+}(t)$ as $\varepsilon \downarrow 0$ where $\dot{\mathbb{R}}_+ = \mathbb{R}_+ \setminus \{0\}$ and $\chi_{\dot{\mathbb{R}}_+}$ is the characteristic function of this set.

For every $n \geqslant 1$, let $\{\eta_k^n\}_{k \geqslant 1} \subseteq C_0^1(Z)$ be such that $\eta_k^n \to x_n$ in $W_0^{1,p}(Z)$. We have

$$\left(\eta_k^n - \psi\right)^+ \in W^{1,p}(Z) \quad \text{(see Evans and Gariepy [5, p. 130]) and}$$
$$\left(\eta_k^n - \psi\right)^+ \to (x_n - \psi)^+ = x_n - \psi \quad \text{in } W_0^{1,p}(Z) \text{ as } k \to \infty \text{ for all } n \geqslant 1$$

(recall that since $x_n \in C$ we have $x_n \geqslant \psi$ for all $n \geqslant 1$). We have $\vartheta_k^n = (\eta_k^n - \psi)^+ + \psi \in W_0^{1,p}(Z)$ and

$$\begin{aligned} \vartheta_k^n &= \left(\eta_k^n - \psi\right)^+ + \psi \to (x_n - \psi) + \psi \\ &= x_n \quad \text{in } W_0^{1,p}(Z) \text{ as } k \to \infty \quad \text{with} \\ \vartheta_k^n &\geqslant \psi \quad \text{for all } k \geqslant 1 \text{ and } n \geqslant 1 \quad \left(\text{i.e., } \vartheta_k^n \in C\right). \end{aligned}$$

So we have just seen that for every $n \geqslant 1$, we can find $\theta_n \in W_0^{1,p}(Z) \cap L^\infty(Z)$ such that

$$\psi \leqslant \vartheta_n \quad \text{and} \quad \|x_n - \vartheta_n\| < \frac{1}{n}.$$

Let $\hat{c} > 0$ be the Poincaré constant, i.e., for every $x \in W_0^{1,p}(Z)$ we have $\|x\| \leqslant \hat{c}\|Dx\|_p$. Set $\hat{\vartheta}_n = \vartheta_n/\|\vartheta_n\|$ and $u_n = x_n + \frac{1}{t}\gamma_\varepsilon(\hat{\vartheta}_n)\hat{\vartheta}_n$ with $t > 2\hat{c}$. We have:

(a) If $\hat{\vartheta}_n(z) \geqslant 0$, then $\gamma_\varepsilon(\hat{\theta}_n(z)) \geqslant 0$ and so $u_n(z) \geqslant x_n(z) \geqslant \psi(z)$.
(b) If $\hat{\vartheta}_n(z) < 0$, then $\gamma_\varepsilon(\hat{\vartheta}_n(z)) = 0$ and so $u_n(z) = x_n(z) \geqslant \psi(z)$.

So it follows that for every $n \geqslant 1$, $u_n \in C$. Using Lemma 1 we have

$$\begin{aligned} D(x_n - u_n) &= \frac{1}{t} D\left(\gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n\right) = \frac{1}{t}\left(\gamma_\varepsilon'\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n + \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\right) D\hat{\vartheta}_n \\ &= \frac{1}{t}\left(1 + \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\right) D\hat{\vartheta}_n \quad \left(\text{since } \gamma_\varepsilon'\left(\hat{\vartheta}_n\right) = \frac{1}{\hat{\vartheta}_n}\right) \end{aligned}$$

$$\Rightarrow \quad \left\| D(x_n - u_n) \right\|_p \leqslant \frac{2}{t} \left\| D\hat{\vartheta}_n \right\|_p \leqslant \frac{2}{t} \quad \left(\text{since } \left\| \hat{\vartheta}_n \right\| = 1\right)$$

$$\Rightarrow \quad \|x_n - u_n\| \leqslant \frac{2\hat{c}}{t}.$$

Since $t > 2\hat{c}$, we infer that $\|x_n - u_n\| < 1$ with $u_n \in C$. Hence we can use u_n as our test function in (5) and obtain

$$\left(1 + \|x_n\|\right)\left\langle x_n^*, x_n - u_n \right\rangle = \frac{1 + \|x_n\|}{t} \left\langle x_n^*, \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n \right\rangle \leqslant \varepsilon_n. \tag{6}$$

Let $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be the nonlinear operator defined by

$$\left\langle A(x), y \right\rangle = \int_Z \|Dx\|^{p-2} (Dx, Dy)_{\mathbb{R}^N} \, dz \quad \text{for all } x, y \in W_0^{1,p}(Z).$$

We know that A is monotone, demicontinuous, hence it is maximal monotone (see Hu and Papageorgiou [6, p. 309]). Also we have

$$x_n^* = A(x_n) - \lambda_1 |x_n|^{p-2} x_n - u_n^*, \quad n \geqslant 1,$$

with $u_n^* \in L^q(Z), u_n^*(z) \in \partial j(z, x_n(z))$ a.e. on Z (see, for example, Hu and Papageorgiou [7, p. 315]). Using this in (6), we obtain

$$\left\langle A(x_n), \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n \right\rangle - \lambda_1 \int_Z |x_n|^{p-2} x_n \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n \, dz - \int_Z u_n^* \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n \, dz$$
$$\leqslant \frac{t\varepsilon_n}{1 + \|x_n\|}.$$

Dividing by $\|x_n\|^{p-1}$ and exploiting the $(p-1)$-homogeneity of A, we have

$$\left\langle A(y_n), \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n \right\rangle - \lambda_1 \int_Z |y_n|^{p-2} y_n \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n \, dz$$
$$- \int_Z \frac{u_n^*}{\|x_n\|^{p-1}} \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n \, dz \leqslant \frac{t\varepsilon_n}{(1 + \|x_n\|)\|x_n\|^{p-1}}. \tag{7}$$

From the definition of A we have

$$\begin{aligned}\left\langle A(y_n), \gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n \right\rangle &= \int_Z \|Dy_n\|^{p-2} \left(Dy_n, D\left(\gamma_\varepsilon\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n\right)\right)_{\mathbb{R}^N} dz \\ &= \int_Z \|Dy_n\|^{p-2} \left(Dy_n, \gamma_\varepsilon\left(\hat{\vartheta}_n\right) D\hat{\vartheta}_n\right)_{\mathbb{R}^N} dz \\ &\quad + \int_Z \|Dy_n\|^{p-2} \left(Dy_n, \gamma_\varepsilon'\left(\hat{\vartheta}_n\right)\hat{\vartheta}_n D\hat{\vartheta}_n\right)_{\mathbb{R}^N} dz\end{aligned}$$

$$= \int_Z \gamma_\varepsilon(\hat{\vartheta}_n)\|Dy_n\|^{p-2}(Dy_n, D\hat{\vartheta}_n)_{\mathbb{R}^N}\, dz + \int_{\{\delta(\varepsilon)\leqslant\hat{\vartheta}_n\leqslant\varepsilon\}} \|Dy_n\|^{p-2}(Dy_n, D\hat{\vartheta}_n)_{\mathbb{R}^N}\, dz. \tag{8}$$

Note that

$$\begin{aligned}\|\hat{\vartheta}_n - y_n\| &= \left\|\frac{\vartheta_n}{\|\vartheta_n} - \frac{x_n}{\|x_n\|}\right\| \leqslant \frac{2\|x_n\|\|\vartheta_n - x_n\|}{\|x_n\|\|\vartheta_n\|} \\ &= \frac{2}{n\|\vartheta_n\|} \to 0 \quad \text{as } n\to\infty, \\ \Rightarrow \quad & \hat{\vartheta}_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(Z) \quad \text{and} \\ & \hat{\vartheta}_n \to y \quad \text{in } L^p(Z) \quad \text{as } n\to\infty.\end{aligned}$$

We have $\hat{\vartheta}_n = y_n + e_n$ with $e_n \in W_0^{1,p}(Z)$, $\|e_n\| \leqslant 1/n$, $n \geqslant 1$. So

$$\int_Z \gamma_\varepsilon(\hat{\vartheta}_n)\|Dy_n\|^{p-2}(Dy_n, D\hat{\vartheta}_n)_{\mathbb{R}^N}\, dz = \int_Z \gamma_\varepsilon(\hat{\vartheta}_n)\|Dy_n\|^p\, dz + \int_Z \gamma_\varepsilon(\hat{\vartheta}_n)\|Dy_n\|^{p-1}\|De_n\|\, dz. \tag{9}$$

From Hölder's inequality, we have

$$\int_Z \gamma_\varepsilon(\hat{\vartheta}_n)\|Dy_n\|^{p-1}\|De_n\|\, dz \leqslant \|Dy_n\|_p^{p-1}\|De_n\|_p \to 0 \quad \text{as } n\to\infty. \tag{10}$$

Also if $f \in L^q(Z,\mathbb{R}^N)$, then $\gamma_\varepsilon(\hat{\vartheta}_n)^{1/p} f \to \gamma_\varepsilon(y)^{1/p} f$ in $L^q(Z,\mathbb{R}^N)$ and since $Dy_n \xrightarrow{w} Dy$ in $L^p(Z,\mathbb{R}^N)$ we have

$$\begin{aligned}&\int_Z \big(Dy_n, \gamma_\varepsilon(\hat{\vartheta}_n)^{1/p} f\big)_{\mathbb{R}^N}\, dz \to \int_Z \big(Dy, \gamma_\varepsilon(y)^{1/p} f\big)_{\mathbb{R}^N}\, dz \\ \Rightarrow \quad & \gamma_\varepsilon(\hat{\vartheta}_n)^{1/p} Dy_n \xrightarrow{w} \gamma_\varepsilon(y)^{1/p} Dy \quad \text{in } L^p(Z,\mathbb{R}^N) \\ \Rightarrow \quad & \int_Z \gamma_\varepsilon(y)\|Dy\|^p\, dz \leqslant \liminf_{n\to\infty} \int_Z \gamma_\varepsilon(\hat{\vartheta}_n)\|Dy_n\|^p\, dz.\end{aligned} \tag{11}$$

Using (10) and (11) in (9), we obtain

$$\int_Z \gamma_\varepsilon(y)\|Dy\|^p\, dz \leqslant \liminf_{n\to\infty} \int_Z \gamma_\varepsilon(\hat{\vartheta}_n)\|Dy_n\|^{p-2}(Dy_n, D\hat{\vartheta}_n)_{\mathbb{R}^N}\, dz. \tag{12}$$

Also we have

$$\int_{\{\delta(\varepsilon)\leqslant\hat{\vartheta}_n\leqslant\varepsilon\}} \|Dy_n\|^{p-2}\big(Dy_n, D\hat{\vartheta}_n\big)_{\mathbb{R}^N}\,dz$$
$$= \int_{\{\delta(\varepsilon)\leqslant\hat{\vartheta}_n\leqslant\varepsilon\}} \|Dy_n\|^p\,dz + \int_{\{\delta(\varepsilon)\leqslant\hat{\vartheta}_n\leqslant\varepsilon\}} \|Dy_n\|^{p-2}(Dy_n, De_n)_{\mathbb{R}^N}\,dz.$$

Because $e_n \to 0$ in $W_0^{1,p}(Z)$, we have

$$\int_{\{\delta(\varepsilon)\leqslant\hat{\vartheta}_n\leqslant\varepsilon\}} \|Dy_n\|^{p-2}(Dy_n, De_n)_{\mathbb{R}^N}\,dz \to 0.$$

In addition recall that $Dy_n \xrightarrow{w} Dy$ in $L^p(Z, \mathbb{R}^N)$ as $n \to \infty$. Also if

$$E_n = \big\{z \in Z\colon\ \delta(\varepsilon) \leqslant \hat{\vartheta}_n(z) \leqslant \varepsilon\big\} \quad \text{and} \quad E^0 = \big\{z \in Z\colon\ \delta(\varepsilon) < y(z) < \varepsilon\big\},$$

then because $\hat{\vartheta}_n(z) \to y(z)$ a.e. on Z, we have $\chi_{E_n}(z) \to \chi_{E^0}(z)$ a.e. on Z. Moreover, from Stampacchia's theorem (see Kesavan [8, p. 256]), we know that $Dy(z) = 0$ a.e. on $\{z \in Z\colon\ y(z) = \delta(\varepsilon)\}$ and $Dy(z) = 0$ a.e. on $\{z \in Z\colon\ y(z) = \varepsilon\}$. Therefore we have

$$\int_{\{\delta(\varepsilon)\leqslant y\leqslant\varepsilon\}} \|Dy\|^p\,dz \leqslant \liminf_{n\to\infty} \int_{\{\delta(\varepsilon)\leqslant\hat{\vartheta}_n\leqslant\varepsilon\}} \|Dy_n\|^p\,dz.$$

So finally we can write that

$$\int_{\{\delta(\varepsilon)\leqslant y\leqslant\varepsilon\}} \|Dy\|^p\,dz$$
$$\leqslant \liminf_{n\to\infty} \int_{\{\delta(\varepsilon)\leqslant\hat{\vartheta}_n\leqslant\varepsilon\}} \|Dy_n\|^{p-2}\big(Dy_n, D\hat{\vartheta}_n\big)_{\mathbb{R}^N}\,dz. \tag{13}$$

Now we return to (8), pass to the limit as $n \to \infty$ and use (12) and (13), to obtain

$$\int_Z \gamma_\varepsilon(y)\|Dy\|^p\,dz + \int_{\{\delta(\varepsilon)\leqslant y\leqslant\varepsilon\}} \|Dy\|^p\,dz$$
$$\leqslant \liminf_{n\to\infty}\big\langle A(y_n), \gamma_\varepsilon\big(\hat{\vartheta}_n\big)\hat{\vartheta}_n\big\rangle. \tag{14}$$

Also since $y_n, \vartheta_n \to y$ in $L^p(Z)$, we have that

$$\lambda_1 \int_Z |y_n|^{p-2} y_n \gamma_\varepsilon\big(\hat{\vartheta}_n\big)\hat{\vartheta}_n\,dz \quad \to \quad \lambda_1 \int_Z \gamma_\varepsilon(y)|y|^p\,dz \quad \text{as } n \to \infty. \tag{15}$$

First suppose that hypotheses $H(j)$(i) → (iv) are in effect. By virtue of hypothesis $H(j)$(iii) we have

$$\frac{|u_n^*(z)|}{\|x_n\|^{p-1}} \leqslant \frac{\alpha(z)}{\|x_n\|^{p-1}} + c\left|y_n(z)\right|^{p-1} \quad \text{a.e. on } Z.$$

So $\{u_n^*/\|x_n\|^{p-1}\}_{n\geqslant 1} \subseteq L^q(Z)$ is bounded and by passing to a subsequence if necessary, we may assume that $u_n^*/\|x_n\|^{p-1} \xrightarrow{w} \xi$ in $L^q(Z)$. Given $\delta > 0$, let

$$Z_{\delta,n} = \left\{z \in Z\colon\ x_n(z) > 0 \text{ and } \frac{u_n^*(z)}{x_n(z)^{p-1}} \leqslant \beta(z) + \delta\right\}.$$

Observe that $x_n(z) \to +\infty$ a.e. on $\{y > 0\}$ and so by virtue of hypothesis $H(j)$(iv) we have that $\chi_{Z_{\delta,n}}(z) \to 1$ a.e. on $\{y > 0\}$. We have

$$\begin{aligned}\chi_{Z_{\delta,n}}(z)\frac{u_n^*(z)}{\|x_n\|^{p-1}} &= \chi_{Z_{\delta,n}}(z)\frac{u_n^*(z)}{x_n(z)^{p-1}}y_n(z)^{p-1}\\ &\leqslant \chi_{Z_{\delta,n}}(z)\big(\beta(z)+\delta\big)y_n(z)^{p-1}.\end{aligned}$$

Passing to the weak limit in $L^q(y > 0)$, we obtain

$$\begin{aligned}&\xi(z) \leqslant \big(\beta(z)+\delta\big)y(z)^{p-1} \quad \text{a.e. on } \{y>0\},\\ &\quad\Rightarrow \quad \xi(z) \leqslant \beta(z)y(z)^{p-1} \quad \text{a.e. on } \{y>0\} \quad (\text{letting } \delta \downarrow 0).\end{aligned}$$

Also note that for almost all $z \in \{y = 0\}$

$$\frac{|u_n^*(z)|}{\|x_n\|^{p-1}} \leqslant \frac{\alpha(z)}{\|x_n\|^{p-1}} + c\left|y_n(z)\right|^{p-1} \to 0 \quad \text{as } n \to \infty,$$

hence $\xi(z) = 0$ a.e. on $\{y = 0\}$. Because $y \geqslant 0$, we have that $\xi(z) \leqslant \beta(z)y(z)^{p-1}$ a.e. on Z.

Therefore if we pass to the limit in (7) and we use (14), (15) and the fact that $u_n^*/\|x_n\|^{p-1} \xrightarrow{w} \xi$ in $L^q(Z)$, we obtain

$$\begin{aligned}&\int_Z \gamma_\varepsilon(y)\|Dy\|^p\,dz + \int_{\{\delta(\varepsilon)\leqslant y\leqslant\varepsilon\}} \|Dy\|^p\,dz - \lambda_1\int_Z \gamma_\varepsilon(y)y^p\,dz\\ &\quad - \int_Z \beta y^p\gamma_\varepsilon(y)\,dz \leqslant 0.\end{aligned}$$

Let $\varepsilon \downarrow 0$. We have

$$\|Dy\|_p^p - \lambda_1\|y\|_p^p - \int_Z \beta y^p\,dz \leqslant 0. \tag{16}$$

Let $k(z) = \lambda_1 + \beta(z)$. Then $k(z) \leqslant \lambda_1$ a.e. on Z with strict inequality on a set of positive measure. We claim that there exists $\gamma > 0$ such that

$$\begin{aligned}\psi(v) &= \|Dv\|_p^p - \int_Z k(z)\big|v(z)\big|^p\,dz \\ &\geqslant \gamma\|Dv\|_p^p \quad \text{for all } v \in W_0^{1,p}(Z). \end{aligned} \tag{17}$$

Note that $\psi \geqslant 0$ (see (3)) and suppose (17) is not true. Then we can find $\{v_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ with $\|Dv_n\|_p = 1$ such that $\psi(v_n) \downarrow 0$. We may assume that $v_n \xrightarrow{w} v$ in $W_0^{1,p}(Z)$ and $v_n \to v$ in $L^p(Z)$. So in the limit we have (assuming without loss of generality that $k \geqslant 0$)

$$\begin{aligned}&\|Dv\|_p^p \leqslant \int_Z k(z)\big|v(z)\big|^p\,dz \leqslant \lambda_1\|v\|_p^p \\ &\quad\Rightarrow \quad v = \pm u_1 \quad \text{or} \quad v = 0 \quad \big(\text{see (3)}\big).\end{aligned}$$

If $v = 0$, then $\|Dv_n\|_p \to 0$ as $n \to \infty$, a contradiction since $\|Dv_n\|_p = 1$ for all $n \geqslant 1$. So $v = \pm u_1$ and we have

$$\|Dv\|_p^p \leqslant \int_Z k(z)\big|v(z)\big|^p\,dz < \lambda_1\|v\|_p^p,$$

a contradiction to (3). So (17) holds. Using this in (16), we obtain

$$\gamma\|Dy\|_p^p \leqslant 0, \quad \text{i.e., } y = 0.$$

Then we have $\limsup\langle A(y_n), y_n\rangle \leqslant 0$ and so $Dy_n \to 0$ in $L^p(Z, \mathbb{R}^N)$, hence $y_n \to 0$ in $W_0^{1,p}(Z)$ (by Poincaré's inequality), a contradiction to the fact that $\|y_n\| = 1$. This proves that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded.

Now suppose that hypotheses $H(j)'$(i) $\to$ (iv) are in effect. Then as before we can verify that $u_n^*/\|x_n\|^{p-1} \xrightarrow{w} \xi_1$ in $L^q(Z)$ and $\xi(z) \leqslant 0$ a.e. on Z. Hence if we pass to the limit in (7) and by sending $\varepsilon \downarrow 0$, we obtain

$$\|Dy\|_p^p = \lambda_1\|y\|_p^p, \quad \Rightarrow \quad y = +u_1 \quad \text{or} \quad y = 0 \quad (\text{recall } y \geqslant 0).$$

Again we cannot have $y = 0$, or otherwise $y_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geqslant 1$. So we must have $y = +u_1$. Then from the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq C$ (see (4)) we have

$$\begin{aligned}&\frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p - \int_Z j\big(z, x_n(z)\big)\,dz \leqslant M_1 \\ &\quad\Rightarrow \quad -M_1 \leqslant \int_Z j\big(z, x_n(z)\big)\,dz \quad \big(\text{see (3)}\big).\end{aligned}$$

Because $y(z) = +u_1(z) > 0$ for all $z \in Z$, we have that $x_n(z) \to +\infty$ a.e. on Z and from hypothesis $H(j)'$(iv) we have that $\int_Z j(z, x_n(z))\,dz \to -\infty$, a contradiction.

Therefore in both cases, we have proved that $\{x_n\}_{n\geqslant 1} \subseteq C$ is bounded in $W_0^{1,p}(Z)$. So by passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in $L^p(Z)$. For $r > 1$, we consider the sequence $\{u_n\}_{n\geqslant 1} \subseteq C$ defined by

$$u_n = \begin{cases} x & \text{if } \|x_n - x\| \leqslant 1, \\ \left(1 - \frac{1}{r\|x_n - x\|}\right)x_n + \frac{1}{r\|x_n - x\|}x & \text{if } \|x_n - x\| \geqslant 1. \end{cases}$$

Evidently for all $n \geqslant 1$, $\|x_n - u_n\| < 1$. Also we have

$$\left\langle x_n^*, x_n - u_n\right\rangle = \begin{cases} \left\langle x_n^*, x_n - x\right\rangle & \text{if } \|x_n - x\| < 1, \\ \left\langle x_n^*, \frac{x_n - x}{r\|x_n - x\|}\right\rangle & \text{if } \|x_n - x\| \geqslant 1. \end{cases}$$

From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subseteq C$ we have

$$\left\langle x_n^*, x_n - u_n\right\rangle \leqslant \varepsilon_n' \quad \text{with } \varepsilon_n' \downarrow 0,$$
$$\Rightarrow \quad \left\langle x_n^*, x_n - x\right\rangle \leqslant \varepsilon_n'' \quad \text{with } \varepsilon_n'' = \max\left\{\varepsilon_n', r\|x_n - x\|\varepsilon_n'\right\} \downarrow 0.$$

Recall that $x_n^* = A(x_n) - \lambda_1|x_n|^{p-2}x_n - u_n^*$ and $\int_Z u_n^*(x_n - x)\,dz \to 0$ (hypothesis $H(j)$(iii) or $H(j)'$(iii)), $\lambda_1\int_Z |x_n|^{p-2}x_n(x_n - x)\,dz \to 0$. So we have $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leqslant 0$. Because A is maximal monotone it is generalized pseudomonotone (see Hu and Papageorgiou [6, p. 365]) and so $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$, hence $\|Dx_n\|_p \to \|Dx\|_p$. Since $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$ and the space $L^p(Z, \mathbb{R}^N)$ being uniformly convex, it has the Kadec–Klee property (see Hu and Papageorgiou [6, p. 28]), we have that $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$ and so $x_n \to x$ in $W_0^{1,p}(Z)$. This proves that φ satisfies the nonsmooth C-condition. □

The next result complements the abstract minimax principles obtained in Kyritsi and Papageorgiou [10].

Proposition 3. *If X is a reflexive Banach space, $C \subseteq X$ is nonempty, closed and convex and $\varphi : C \to \mathbb{R}$ is locally Lipschitz, bounded below and satisfies the nonsmooth C-condition on C, then there exists $x \in C$ such that $\varphi(x) = \inf_C \varphi$ and x is a critical point of φ on C.*

Proof. Let $\{x_n\}_{n\geqslant 1} \subseteq C$ be such that $\varphi(x_n) \downarrow \inf_C \varphi$. Invoking Theorem 1.1 of Zhong [25] (with $x_0 = 0$, $\varepsilon = 1/n^2$, $\lambda = 1/\varepsilon$), we obtain a sequence $\{y_n\}_{n\geqslant 1} \subseteq C$ such that for all $n \geqslant 1$

$$\varphi(y_n) \leqslant \varphi(x_n) \quad \left(\text{hence } \varphi(y_n) \downarrow \inf_C \varphi\right), \quad \|y_n\| \leqslant \|x_n\| + \bar{r}$$
$$\text{and} \quad \varphi(v) \geqslant \varphi(y_n) - \frac{1}{n(1 + \|y_n\|)}\|v - y_n\| \quad \text{for all } v \in C,$$

where $\bar{r} > 0$ is such that

$$\int_{\|y_n\|}^{\|y_n\|+\bar{r}} \frac{1}{1+r}\,dr \geqslant \frac{1}{n}.$$

For every $v \in X$ and $t \in (0,1)$ we have

$$\begin{aligned}
&-\frac{t\|v-y_n\|}{n(1+\|y_n\|)} \leqslant \varphi\big(y_n+t(v-y_n)\big)-\varphi(y_n)+\delta_C\big(y_n+t(v-y_n)\big)\\
&\quad\Rightarrow\quad -\|v-y_n\| \leqslant n\big(1+\|y_n\|\big)\\
&\qquad\qquad \times\left(\frac{\varphi(y_n+t(v-y_n))-\varphi(y_n)}{t}+\frac{\delta_C(y_n+t(v-y_n))}{t}\right).
\end{aligned}$$

Note that δ_C is convex (see Section 2) and $\delta_C(y_n)=0$. So

$$\frac{\delta_C(y_n+t(v-y_n))}{t} \leqslant \delta_C(y_n+v-y_n)=\delta_C(v).$$

Also we have

$$\limsup_{t\downarrow 0}\frac{\varphi(y_n+t(v-y_n))-\varphi(y_n)}{t} \leqslant \varphi^0(y_n;v-y_n).$$

Set $v=y_n+h$, $h\in X$, $\eta_n(h)=\varphi^0(y_n;h)$ and $\vartheta_n(h)=\delta_C(y_n+h)$. We have that $(\eta_n+\vartheta_n)(\cdot)$ is a proper, convex and lower semicontinuous function on X with values in $\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ (i.e., $\eta_n+\vartheta_n\in\Gamma_0(X)$ for all $n\geqslant 1$, see Hu and Papageorgiou [6, p. 341]) and $(\eta_n+\vartheta_n)(0)=0$. Since

$$-\|h\| \leqslant n\big(1+\|y_n\|\big)(\eta_n+\vartheta_n)(h)\quad\text{for all } h\in X$$

we can apply Lemma 1.3 of Szulkin [23] and obtain $\bar{u}_n^*\in X^*$, $\|\bar{u}_n^*\|\leqslant 1$ such that

$$\big(\bar{u}_n^*,h\big) \leqslant n\big(1+\|y_n\|\big)(\eta_n+\vartheta_n)(h)\quad\text{for all } h\in X. \tag{18}$$

Since $\eta_n+\vartheta_n\in\Gamma_0(X)$ and $(\eta_n+\vartheta_n)(0)=0$, from (18) it follows that

$$\bar{u}_n^*\in n\big(1+\|y_n\|\big)\partial(\eta_n+\vartheta_n)(0).$$

But because η_n is continuous, convex (in fact sublinear), we have $\partial(\eta_n+\vartheta_n)(0)=\partial\eta_n(0)+\partial\vartheta_n(0)$ (see Hu and Papageorgiou [6, p. 349]). Observe that $\partial\eta_n(0)=\partial\varphi(y_n)$, where the second subdifferential is in the sense of Clarke (see Section 2). Also $\partial\vartheta_n(0)=\partial\delta_C(y_n)=N_C(y_n)$. Therefore we have $\bar{u}_n^*=\bar{v}_n^*+\bar{w}_n^*$ with $\bar{v}_n^*\in n(1+\|y_n\|)\partial\varphi(y_n)$ and $\bar{w}_n^*\in n(1+\|y_n\|)N_C(y_n)$. Set

$$v_n^*=\frac{1}{n(1+\|y_n\|)}\bar{v}_n^*\in\partial\varphi(x_n)\quad\text{and}\quad w_n^*=\frac{1}{n(1+\|y_n\|)}\bar{w}_n^*\in N_C(y_n).$$

From the definition of the normal cone (see Section 2), we have

$$\big(w_n^*,y_n-v\big)\geqslant 0\quad\text{for all } v\in C. \tag{19}$$

Recalling the definition of m_C, we have

$$\begin{aligned}
&\big(1+\|y_n\|\big)m_C(y_n)\\
&\quad\leqslant \big(1+\|y_n\|\big)\sigma\big(v_n^*, C_1(y_n)\big)\\
&\quad= \big(1+\|y_n\|\big)\sup\big[\big(u_n^*, y_n-v\big)-\big(w_n^*, y_n-v\big)\colon\ v\in C,\ \|y_n-v\|<1\big]\\
&\quad\leqslant \big(1+\|y_n\|\big)\sigma\big(u_n^*, C_1(y_n)\big)\quad\big(\text{see (19)}\big)\\
&\quad\leqslant \frac{1}{n}\quad\left(\text{recall that } u_n^*=\frac{1}{n(1+\|y_n\|)}\bar{u}_n^* \text{ and } \|u_n^*\|\leqslant 1\right)\\
&\quad\Rightarrow\quad \big(1+\|y_n\|\big)m_C(y_n)\to 0\quad \text{as } n\to\infty.
\end{aligned}$$

But by hypothesis φ satisfies the nonsmooth C-condition on C. So we may assume that $y_n\to x$. Then $\varphi(x)=\inf_C\varphi$ and from Clarke [4, p. 52] we know that

$$0\in\partial\varphi(x)+N_C(x).$$

So there exists $x^*\in N_C(x)$ such that $-x^*\in\partial\varphi(x)$. We have $\langle x^*, x-v\rangle\geqslant 0$ for all $v\in C$, hence $\langle -x^*, x-v\rangle\leqslant 0$ for all $v\in C$ and so $m_C(x)\leqslant 0$. But $m_C\geqslant 0$. Therefore $m_C(x)=0$ and so $x\in C$ is a critical point of φ on C. $\square$

4. Existence theorems

In this section we use the auxiliary results of Section 3 to prove two existence theorems for the obstacle problem.

Theorem 4. *If hypotheses $H(j)$ or $H(j)'$ hold and $\psi\in W^{1,p}(Z)\cap L^\infty(Z)$ with $\psi|_\Gamma\leqslant 0$, then problem* (1) *has a solution $x\in C$.*

Proof. For $x\in C$, using (3) and hypothesis $H(j)$(v) or $H(j)'$(v), we have

$$\begin{aligned}
&\varphi(x)\geqslant -\int_Z j\big(z,x(z)\big)\,dz\geqslant -\int_Z g(z)\,dz\\
&\quad\Rightarrow\quad \varphi \text{ is bounded below on } C.
\end{aligned}$$

From Proposition 2 we know that φ satisfies the nonsmooth C-condition on C. So we can apply Proposition 3 and obtain $x\in C$ such that $\varphi(x)=\inf_C\varphi$. Then as in the proof of Proposition 3 we obtain $x^*\in\partial\varphi(x)$ such that $(x^*, y-x)\geqslant 0$ for all $y\in C$. But $x^*=A(x)-\lambda_1|x|^{p-2}x-u^*$ with $u^*\in L^q(Z)$, $u^*(z)\in\partial j(z,x(z))$ a.e. on Z. Therefore

$$\int_Z \|Dx\|^{p-2}(Dx, Dy-Dx)_{\mathbb{R}^N}\,dz-\lambda_1\int_Z |x|^{p-2}x(y-x)\,dz$$

$$- \int_Z u^*(y-x)\,dz \geqslant 0 \quad \text{for all } y \in C$$

$$\Rightarrow \quad x \in C \text{ is a solution of (1).} \qquad \square$$

By strengthening our hypotheses on the obstacle ψ, we can have a pointwise interpretation of the obstacle problem. Namely consider the following nonlinear hemivariational inequality at resonance with a unilateral constraint:

$$\left\{ \begin{array}{l} -\operatorname{div}\bigl(\|Dx(z)\|^{p-2} Dx(z)\bigr) \\ \quad -\lambda_1 \bigl|x(z)\bigr|^{p-2} x(z) \in \partial j\bigl(z, x(z)\bigr) \quad \text{a.e. on } Z \\ x|_\Gamma = 0, \quad \psi(z) \leqslant x(z) \quad \text{a.e. on } Z \end{array} \right\}. \tag{20}$$

We can solve (20) if we assume that ψ is a kind of lower solution (subsolution) for the Dirichlet hemivariational inequality at resonance. More precisely we make the following assumption.

H_0: $\psi \in W^{1,p} \cap L^\infty(Z)$ with $\psi|_\Gamma \leqslant 0$ and there exists $\underline{u}^* \in L^q(Z)$ with $\underline{u}^*(z) = \min[u^* \in \partial j(z, \psi(z))$ a.e. on $Z]$ such that for all $\vartheta \in C_0^\infty(Z)$ with $\vartheta \geqslant 0$ we have

$$\int_Z \|D\psi\|^{p-2} (D\psi, D\vartheta)_{\mathbb{R}^N}\,dz - \lambda_1 \int_Z |\psi|^{p-2}\psi\vartheta\,dz \leqslant \int_Z \underline{u}^*\vartheta\,dz.$$

Remark. If $j(z,\cdot) \in C^1$, then $\partial j(z,x)$ is a singleton and so the above hypothesis says that for the "smooth" problem ψ is a lower solution. For the multivalued situation (as is ours), usually in the definition of lower solution $\underline{u}^*(z) = \min[u^*\colon u^* \in \partial j(z, \psi(z))]$. Using the Yankov–von Neumann–Aumann projection theorem (see Hu and Papageorgiou [6, p. 149]), one can show that $\underline{u}^*(\cdot)$ is measurable and in fact by virtue of hypothesis $H(j)$(iii) or $H(j)'$(iii), $\underline{u}^* \in L^q(Z)$ and of course $\underline{u}^*(z) \in \partial j(z, \psi(z))$ a.e. on Z.

Theorem 5. *If hypotheses $H(j)$ or $H(j)'$ hold, then problem* (20) *has a solution $x \in C$.*

Proof. Because of Propositions 2 and 3, we know that there exists $x \in C$ such that $m_C(x) = 0$. We can find $x^* \in \partial\varphi(x)$ such that $m_C(x) = \sigma(x^*, C_1(x))$ (see the proof of Proposition 2). Then from the definition of m_C we have

$$\langle x^*, x-u\rangle \leqslant m_C(x) = 0 \quad \text{for all } u \in C \text{ with } \|x-u\| < 1. \tag{21}$$

Let $\vartheta \in W_0^{1,p}(Z)$ and $\varepsilon > 0$ be given and set $u_\varepsilon = (x+\varepsilon\vartheta) + (x+\varepsilon\vartheta-\psi)^- \in W_0^{1,p}(Z)$ (recall $\psi|_\Gamma \leqslant 0$).

We have

$$u_\varepsilon(z) = \begin{cases} \psi(z) & \text{if } (x+\varepsilon\vartheta)(z) \leqslant \psi(z), \\ (x+\varepsilon\vartheta)(z) & \text{if } (x+\varepsilon\vartheta)(z) > \psi(z), \end{cases}$$

hence $u_\varepsilon \in C$.

If $\|x - u_\varepsilon\| < 1$, we have $\langle x^*, x - u_\varepsilon\rangle \leqslant 0$.

If $\|x - u_\varepsilon\| \geqslant 1$, we set

$$y_\varepsilon = \left(1 - \frac{1}{t\|x - u_\varepsilon\|}\right)x + \frac{1}{t\|x - u_\varepsilon\|}u_\varepsilon \in C \quad \text{with } t > 1.$$

We have $x - y_\varepsilon = (x - u_\varepsilon)/(t\|x - u_\varepsilon\|)$ and so $\|x - y_\varepsilon\| = 1/t < 1$. So we can set

$$v_\varepsilon = \begin{cases} u_\varepsilon & \text{if } \|x - u_\varepsilon\| < 1, \\ y_\varepsilon & \text{if } \|x - u_\varepsilon\| \geqslant 1, \end{cases}$$

and have that $v_\varepsilon \in C$ and $\|x - v_\varepsilon\| < 1$. Thus we can use v_ε as our test function in (21) and obtain

$$\begin{aligned}
&\langle x^*, x - v_\varepsilon\rangle \leqslant 0 \\
&\quad \Rightarrow \quad \langle x^*, v_\varepsilon - x\rangle \geqslant 0 \quad \text{for all } \varepsilon > 0 \\
&\quad \Rightarrow \quad 0 \leqslant \begin{cases} \langle x^*, u_\varepsilon - x\rangle & \text{if } \|x - u_\varepsilon\| < 1, \\ \frac{1}{t\|x-u_\varepsilon\|}\langle x^*, u_\varepsilon - x\rangle & \text{if } \|x - u_\varepsilon\| \geqslant 1 \end{cases} \\
&\quad \Rightarrow \quad 0 \leqslant \langle x^*, u_\varepsilon - x\rangle \\
&\quad \Rightarrow \quad -\big\langle x^*, (x + \varepsilon\vartheta - \psi)^-\big\rangle \leqslant \varepsilon\langle x^*, \vartheta\rangle.
\end{aligned} \tag{22}$$

Recall that $x^* = A(x) - \lambda_1|x|^{p-2}x - u^*$ with $u^* \in L^q(Z)$, $u^*(z) \in \partial j(z, x(z))$ a.e. on Z.

Set $x_\varepsilon = (x + \varepsilon\vartheta - \psi)^- \in W_0^{1,p}(Z)$ (since $\psi|_\Gamma \leqslant 0$ and so $x + \varepsilon\vartheta - \psi|_\Gamma \geqslant 0$) and $\underline{x}^* = A(\psi) - \lambda_1|\psi|^{p-2}\psi - \underline{u}^*$. We have

$$-\langle x^*, x_\varepsilon\rangle = -\langle x^* - \underline{x}^*, x_\varepsilon\rangle - \langle \underline{x}^*, x_\varepsilon\rangle.$$

Because of hypothesis H_0 and since $x_\varepsilon \geqslant 0, x_\varepsilon \in W_0^{1,p}(Z)$, we have that $-\langle \underline{x}^*, x_\varepsilon\rangle \geqslant 0$ and so we obtain

$$-\langle x^* - \underline{x}^*, x_\varepsilon\rangle \leqslant -\langle x^*, x_\varepsilon\rangle. \tag{23}$$

Set $Z_-^\varepsilon = \{z \in Z\colon\ (x + \varepsilon\vartheta)(z) < \psi(z)\}$. Exploiting the monotonicity of A, we have

$$\begin{aligned}
-\big\langle A(x) - A(\psi), x_\varepsilon\big\rangle &= -\big\langle A(x) - A(\psi), (x + \varepsilon\vartheta - \psi)^-\big\rangle \\
&= \int_{Z_-^\varepsilon} \big(\|Dx\|^{p-2}(Dx, Dx + \varepsilon D\vartheta - D\psi)_{\mathbb{R}^N} \\
&\quad - \|D\psi\|^{p-2}(D\psi, Dx + \varepsilon D\vartheta - D\psi)_{\mathbb{R}^N}\big)\,dz
\end{aligned}$$

$$\geqslant \varepsilon \int\limits_{Z_-^\varepsilon} \|Dx\|^{p-2}(Dx, D\vartheta)_{\mathbb{R}^N}\, dz$$
$$- \varepsilon \int\limits_{Z_-^\varepsilon} \|D\psi\|^{p-2}(D\psi, D\vartheta)_{\mathbb{R}^N}\, dz. \tag{24}$$

Also we have

$$\lambda_1 \int\limits_Z \big(|x|^{p-2}x - |\psi|^{p-2}\psi\big)(x + \varepsilon\vartheta - \psi)^-\, dz$$
$$= -\lambda_1 \int\limits_{Z_-^\varepsilon} \big(|x|^{p-2}x - |\psi|^{p-2}\psi\big)(x + \varepsilon\vartheta - \psi)\, dz \geqslant 0. \tag{25}$$

The last inequality follows from the fact that $|x|^{p-2}x \geqslant |\psi|^{p-2}\psi$ a.e. on Z since $x \geqslant \psi$ a.e. on Z (recall that $r \to |r|^{p-2}r$ is strictly monotone on $\mathbb{R}$) and because $x + \varepsilon\vartheta < \psi$ a.e. on Z_-^ε.

Note that on Z_-^ε we have $(x + \varepsilon\vartheta)(z) < \psi(z)$, hence $\varepsilon\vartheta(z) < \psi(z) - x(z) \leqslant 0$ a.e. on Z_-^ε and so $\vartheta(z) < 0$ for almost all $z \in Z_-^\varepsilon$. Moreover, by virtue of hypothesis $H(j)$(iii) (respectively $H(j)'$(iii)) we have that $|u^*(z) - \underline{u}^*(z)| \leqslant \eta(z)$ a.e. on Z with $\eta \in L^q(Z)$. On Z_-^ε we have $(x + \varepsilon\vartheta)(z) < \psi(z)$, hence $(x - \psi)(z) < -\varepsilon\vartheta(z)$ and so $\varepsilon\eta(z)\vartheta(z) < -\eta(z)(x - \psi)(z)$. So we can write that

$$\int\limits_Z (u^* - \underline{u}^*)(x + \varepsilon\vartheta - \psi)^-\, dz$$
$$\geqslant \int\limits_{\widehat{Z}_-^\varepsilon} (\underline{u}^* - u^*)(x + \varepsilon\vartheta - \psi)\, dz \quad \big(\widehat{Z}_-^\varepsilon = \{x > \psi,\ x + \varepsilon\vartheta < \psi\}\big)$$
$$= \int\limits_{\widehat{Z}_-^\varepsilon} (\underline{u}^* - u^*)(x - \psi)\, dz + \varepsilon \int\limits_{\widehat{Z}_-^\varepsilon} (\underline{u}^* - u^*)\vartheta\, dz$$
$$\geqslant \int\limits_{\widehat{Z}_-^\varepsilon} -\eta(x - \psi)\, dz + \varepsilon \int\limits_{\widehat{Z}_-^\varepsilon} \eta\vartheta\, dz$$
$$\big(\text{since } \vartheta(z) < 0 \text{ a.e. on } Z_-^\varepsilon,\ \text{see above}\big)$$
$$\geqslant 2\varepsilon \int\limits_{\widehat{Z}_-^\varepsilon} \eta\vartheta\, dz$$
$$\big(\text{since } -\eta(x - \psi) > \varepsilon\eta\vartheta \text{ a.e. on } Z_-^\varepsilon,\ \text{see above}\big). \tag{26}$$

Using (24)–(26) in (23), we have

$$-\langle x^*, x_\varepsilon\rangle \geqslant \varepsilon \int_{Z_-^\varepsilon} \|Dx\|^{p-2}(Dx, D\vartheta)_{\mathbb{R}^N}\,dz - \varepsilon \int_{Z_-^\varepsilon} \|D\psi\|^{p-2}(D\psi, D\vartheta)_{\mathbb{R}^N}\,dz + 2\varepsilon \int_{\widehat{Z}_-^\varepsilon} \eta\vartheta\,dz.$$

Using this inequality in (22) (recall $x_\varepsilon = (x + \varepsilon\vartheta - \psi)^-$) we have

$$\varepsilon\langle x^*, \vartheta\rangle \geqslant \varepsilon \int_{Z_-^\varepsilon} \|Dx\|^{p-2}(Dx, D\vartheta)_{\mathbb{R}^N}\,dz - \varepsilon \int_{Z_-^\varepsilon} \|D\psi\|^{p-2}(D\psi, D\vartheta)_{\mathbb{R}^N}\,dz + 2\varepsilon \int_{\widehat{Z}_-^\varepsilon} \xi\vartheta\,dz.$$

Divide by $\varepsilon > 0$, let $\varepsilon \downarrow 0$ and note that $|\widehat{Z}_-^\varepsilon| \to 0$ (here by $|\cdot|$ we denote the Lebesgue measure on $\mathbb{R}^N$). So we obtain

$$\langle x^*, \vartheta\rangle \geqslant 0 \quad \text{for all } \vartheta \in W_0^{1,p}(Z) \quad \Rightarrow \quad x^* = 0.$$

So we have $A(x) - \lambda_1|x|^{p-2}x = u^*$. Let $\mu \in C_0^\infty(Z)$. From the definition of distributional derivative and since $-\operatorname{div}(\|Dx\|^{p-2}Dx) \in W^{-1,q}(Z)$ (recall the representation of elements in $W^{-1,q}(Z) = W_0^{1,p}(Z)^*$, see, for example, Showalter [20, p. 54]), we obtain

$$\big\langle -\operatorname{div}\big(\|Dx\|^{p-2}Dx\big), \mu\big\rangle - \lambda_1 \int_Z |x|^{p-2}x\mu\,dz = \int_Z u^*\mu\,dz.$$

Since $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$, we conclude that

$$\left\{\begin{array}{l} -\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) - \lambda_1\big|x(z)\big|^{p-2}x(z) \\ \quad = u^*(z) \in \partial j\big(z, x(z)\big) \quad \text{a.e. on } Z, \\ x|_\Gamma = 0, \quad \psi(z) \leqslant x(z) \quad \text{a.e. on } Z \end{array}\right\}$$
$$\Rightarrow \quad x \text{ is a solution of (20).} \qquad \square$$

Remark. The approach of this paper can also be used to study hemivariational inequalities at resonance with other types of constraints, i.e., different closed convex sets C. We will examine these variational–hemivariational inequalities in a forthcoming paper.

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